SpeedLabs

MATHS

CBSE 12th

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1. Let f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} be given by f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}. Write down gof.

Ans. The functions f: {1, 3, 4} → {1, 2, 5} and g: {1, 2, 5} → {1, 3} are defined as

f = {(1, 2), (3, 5), (4, 1)} and g = {(1, 3), (2, 3), (5, 1)}.

$\text{gof}(1) = g(f(1)) = g(2) = 3 \qquad [f(1) = 2 \text{ and } g(2) = 3]$

$\text{gof}(3) = g(f(3)) = g(5) = 1 \qquad [f(3) = 5 \text{ and } g(5) = 1]$

$\text{gof}(4) = g(f(4)) = g(1) = 3 \qquad [f(4) = 1 \text{ and } g(1) = 3]$

∴ gof = {(1,3), (3,1), (4,)3)}

2. Let f, g and h be functions from **R** to **R**. Show that

$(f + g)oh = foh + goh$

$(f.g)\, oh = (foh).(goh)$

Ans. To prove:

$(f + g)\, oh = foh + goh$

Consider:

$((f + g)oh)(x)$

$= (f + g)(h(x)$

$= f(h(x)) + g(g(x))$

$= (foh)(x) + (goh)(x)$

$= \{(foh) + (goh)\}(x)$

$((f + g)oh)(x) = \{(foh) + (goh)\}(x) \qquad \forall x\, \mathbf{R}$

Hence $(f + g)\, oh = foh + goh$.

To prove:

$(f \cdot g)oh = (foh)\cdot(goh)$

Consider:

$$\left(\left(f\cdot g\right)oh\right)\left(x\right)$$

$$= \left(f\cdot g\right)\left(h\left(x\right)\right)$$

$$= f\left(h\left(x\right)\right)\cdot g\left(h\left(x\right)\right)$$

$$= \left(foh\right)\left(x\right)\cdot\left(goh\right)\left(x\right)$$

$$= \left\{\left(foh\right)\left(goh\right)\right\}\left(x\right)$$

$$\therefore \left(\left(f\cdot g\right)oh\right)\left(x\right) = \left\{\left(goh\right)\cdot\left(goh\right)\right\}\left(x\right) \forall x\in R$$

Hence, $\left(f\cdot g\right)oh = \left(foh\right)\cdot\left(goh\right)$.

3. Find gof and fog, if

$\left(i\right)\ f\left(x\right) = \left|x\right|$ and $g\left(x\right) = \left|5x - 2\right|$

$\left(ii\right)\ f\left(x\right) = 8x^3$ and $g\left(x\right) = x^{\frac{1}{3}}$

Ans. $\left(i\right)\ f\left(x\right)=\left|x\right|$ and $g\left(x\right)=\left|5x-2\right|$

$$\therefore \left(gof\right)\left(x\right)=g\left(f\left(x\right)\right)=g\left(|x|\right)=\left|5\left|x\right|-2\right|$$

$$\left(fog\right)\left(x\right)=f\left(g\left(x\right)\right)= f\left(|5x-2|\right)=\left|\left|5x - 2\right|\right|=\left|5x - 2\right|$$

$\left(ii\right)\ f\left(x\right) = 8x^3$ and $g\left(x\right) = x^{\frac{1}{3}}$

$$\therefore \left(gof\right)\left(x\right)= g\left(f\left(x\right)\right)= g\left(8x^3\right) = \left(8x^3\right)^{\frac{1}{3}}=2x$$

$$\left(fog\right)\left(x\right)= f\left(g\left(x\right)\right)=f\left(x^{\frac{1}{3}}\right)=8x\left(x^{\frac{1}{3}}\right)^3=8x$$

4. If $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$, show that f o f(x) = x, for all $x\neq\frac{2}{3}$. What is the inverse of f?

Ans. It is given that $f(x)=\frac{(4x+3)}{(6x-4)}, x\neq\frac{2}{3}$

$$(fof)(x) = f\left(f(x)\right) = f\left(\frac{4x+3}{6x-4}\right)$$

$$= \frac{\left(\frac{4x+3}{6x-4}\right)+3}{\left(\frac{6x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16} = \frac{34x}{34} = x$$

Therefore, $fof\ (x) = x$, for all $x \neq \frac{2}{3}$.

$\Rightarrow fof = I$

Hence, the given function f is invertible and the inverse of f is f itself.

5. State with reason whether following functions have inverse

(i) f: {1, 2, 3, 4} → {10} with

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} with

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} with

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

Ans. (i) f: {1, 2, 3, 4} → {10}defined as:

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of f, we can see that f is a many one function as: f(1) = f(2) = f(3) = f(4) = 10

∴f is not one-one.

Hence, function f does not have an inverse.

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

From the given definition of g, it is seen that g is a many one function as: g(5) = g(7) = 4.

∴g is not one-one,

Hence, function g does not have an inverse.

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} defined as:

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under h.

∴Function h is one-one.

Also, h is onto since for every element y of the set {7, 9, 11, 13}, there exists an element x in the set {2, 3, 4, 5} such that h(x) = y.

Thus, h is a one-one and onto function. Hence, h has an inverse.

6. Show that f: [−1, 1] → **R**, given by is $f(x) = \frac{x}{(x+2)}$ one-one. Find the inverse of the function f: [−1, 1] → Range f.

(Hint: For y ∈Range f, y = $f(x)=\frac{x}{(x+2)}$ for some x in [−1, 1], i.e., $f(x)=\frac{x}{(x+2)}$

Ans. f: [−1, 1] → R is given as $f(x)=\frac{x}{(x+2)}$

f: [−1, 1] → R is given as

Let f(x) = f(y).

$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2}$

$\Rightarrow xy+2x=xy+2y$

$\Rightarrow 2x=2y$

$\Rightarrow x=y$

∴ f is a one-one function.

It is clear that f: [−1, 1] → Range f is onto.

∴ f: [−1, 1] → Range f is one-one and onto and therefore, the inverse of the function:

f: [−1, 1] → Range f exists.

Let g: Range f → [−1, 1] be the inverse of f.

Let y be an arbitrary element of range f.

Since f: [−1, 1] → Range f is onto, we have:

y = f (x)for same x ∈ [−1, 1]

$\Rightarrow y=\frac{x}{x+2}$

$\Rightarrow xy+2y=x$

$\Rightarrow x(1-y)=x$

$\Rightarrow x=\frac{2y}{1-y}, y\neq 1.$

Now, let us define g: Range f → [−1, 1] as

$g\,(y)=\frac{2y}{1-y}, y\neq 1.$

Now, $(gof)(x)=g\big(f(x)\big)=g\left(\frac{2y}{1-y}\right)=\frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}}=\frac{2x}{x+2-x}=\frac{2x}{2}=x$

$(fog)(y)=f\big(g(y)\big)=f\left(\frac{2y}{1-y}\right)=\frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=y$

$\therefore gof = I_{[-1,1]}$ and $fog = I_{\text{Range f}}$

$\Rightarrow f^{-1}(y)=\frac{2y}{1-y}, y \neq 1$

7. Consider f: R → R given by f(x) = 4x + 3. Show that f is invertible. Find the inverse of f.

Ans. f: **R → R** is given by,

$f(x) = 4x + 3$

One-one:

Let $f(x) = f(y)$.

$\Longrightarrow 4x + 3 = 4y + 3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

∴ f is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$\Rightarrow x = \dfrac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in R$, there exists $x = \dfrac{y-3}{4} \in \mathbf{R}$ such that

$f(x) = f\left(\dfrac{y-3}{4}\right) = 4\left(\dfrac{y-3}{4}\right) + 3 = y.$

∴ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: **R→ R** by $(g) = \dfrac{y-3}{4}$

$$\text{Now,}(gof)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(fo\,g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = 7$$

∴ $gof = fog = I_R$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \dfrac{y-3}{4}.$

8. Consider f: $\mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where R+ is the set of all non-negative real numbers.

Ans. f: R+ → [4, ∞) is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y$ [as $x = y \in \mathbf{R}$,]

$\therefore$ f is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$\Rightarrow x^2 = y - 4 \geq 0$ [as $y \geq 4$]

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in R$ such that

$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y.$

$\therefore$ f is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \rightarrow \mathbf{R}+$ by,

Now. $gof(x) = g(f(x)) = g(x^2+4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$

And. $fog(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$

$\therefore gof = fo\,g = I_{R+}$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \sqrt{y-4}$.

9. Consider f: $\mathbf{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(y+6)-1}{3}\right)$

Ans. f: $\mathbf{R}_+ \rightarrow [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$

$\Rightarrow (3x+1)^2 = y + 6$

$\Rightarrow 3x+ = \sqrt{y+1}$ [as $y \geq -5 \Rightarrow y + 6 > 0$]

$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$

$\therefore$f is onto, thereby range $f = [-5, \infty)$.

Let us define g: $[-5, \infty) \rightarrow \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$

We now have:

$(gof)(x) = g(f(x)) = g(9x^2+6x-5)$

$=g((3x+1)^2-6)$

$=\dfrac{\sqrt{(3x+1)^2-6+6}-1}{3}$

$=\dfrac{3x+1-1}{3}=x$

And, $(fog)(y) = f(g(y)) = f=\left(\dfrac{\sqrt{y+6}-1}{3}\right)$

$=\left[3\left(\dfrac{\sqrt{y+6}-1}{3}\right)+1\right]^2-6$

$=\left(\sqrt{y+6}\right)^2-6=y+6-6=y$

$\therefore$ gof $= I_{\mathbf{R}}$, and fo g $= I\,[-5,\infty)$

Hence, f is invertible and the inverse of f is given by

$f^{-1}\,(y)\,g\,(y) = \dfrac{\sqrt{y+6}-1}{3}$.

10. Let f: X → Y be an invertible function. Show that f has unique inverse.

(Hint: suppose g^1 and g^2 are two inverses of f. Then for all y ∈ Y,

$fog_1\ (y) = I_y\ (y) = fog_2\ (y)$. Use one-one ness of f).

Ans. Let f: X → Y be an invertible function.

Also, suppose f has two inverses (say t_1 and g_2).

Then, for all y ∈Y, we have:

$fog_1(y) = I_y(y) = fog_2(y)$

$\Rightarrow\ f\left(g_1(y)\right) = f\,(g_2(y))$

$\Rightarrow g_1(y) = g_2(y)$ [f is invertible $\Rightarrow$ f is one − one]

$\Rightarrow g_1 = g_2$ [g is one − one]

Hence, f has a unique inverse.

11. Consider f: {1, 2, 3} → {a, b, c} given by f(1) = a, f(2) = b and f(3) = c. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Ans. Function f: {1, 2, 3} → {a, b, c} is given by,

f(1) = a, f(2) = b, and f(3) = c

If we define g: {a, b, c} → {1, 2, 3} as g(a) = 1, g(b) = 2, g(c) = 3, then we have:

$(fog)(a) = f(g(a)) = f(1) = a$

$(fog)(b) = f(g(b)) = f(2) = b$

$(fog)(c) = f(g(c)) = f(3) = c$

And,

$(gof)(1) = g(f(1)) = g(a) = 1$

$(gof)(2) = g(f(2)) = g(b) = 2$

$(gof)(3) = g(f(3)) = g(c) = 3$

$\therefore\ gof = I_x$ and $fog = I_y$, where X = {1, 2, 3} and Y= {a, b, c}.

Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}$: {a, b, c} → {1, 2, 3} is given by,

$f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of g.

If we define h: {1, 2, 3} → {a, b, c} as

h(1) = a, h(2) = b, h(3) = c, then we have:

$(goh)(1) = g(h(1)) = g(a) = 1$

$(goh)(2) = g(h(2)) = g(b) = 2$

$(goh)(3) = g(h(3)) = g(c) = 3$

And

$(hog)(a) = h((a)) = h(1) = a$

$(hog)(b) = h((b)) = h(2) = b$

$(hog)(c) = h((c)) = h(3) = c$

$\therefore\ goh\,I_x$ and $hog\,I_y$, where X = {1, 2, 3} and Y = {a, b, c}.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that h = f.

Hence, $(f^{-1})^{-1} = f$.

12. Let f: X → Y be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

Ans. Let f: X → Y be an invertible function.

Then, there exists a function g: Y → X such that gof = IXand fog = IY.

Here, f−1 = g.

Now, $gof = I_X$and $fog = I_Y$

$\Rightarrow f^{-1}of = I_X$ and $fof^{-1}= I_Y$

Hence, f−1: Y → X is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

13. If f: R $\rightarrow$ R be given by $f(x)=(3-x^3)^{\frac{1}{3}}$, then fof(x) is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3-x^3)$

Ans. f: **R** $\rightarrow$ **R** is given as $f\left(x\right)=\left(3-x^3\right)^{\frac{1}{3}}$

$$\therefore fof(x)=f(f(x))=f\left((3-x^3)^{\frac{1}{3}}\right)=\left[3-\left(\left(3-x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$=\left[3-\left(3-x^3\right)\right]^{\frac{1}{3}}=(x^3)^{\frac{1}{3}}=x$$

$$\therefore fof(x)=x$$

The correct answer is C.

14. Let $f:\mathbf{R}-\left\{-\frac{4}{3}\right\}\rightarrow\mathbf{R}$ be a function defined as $f(x)=\frac{4x}{3x+4}$. The inverse of f is map g:

Range f: $\mathbf{R}-\left\{-\frac{4}{3}\right\}\rightarrow$ given by

(A) $g(y)=\frac{3y}{3-4y}$ (B) $g(y)=\frac{4y}{4-3y}$

(C) $g(y)=\frac{4y}{3-4y}$ (D) $g(y)=\frac{3y}{4-3y}$

Ans. It is given that $f:R-\left\{-\frac{4}{3}\right\}\rightarrow R$ is defined as $f(x)=\frac{4x}{3x+4}$.

Let y be an arbitrary element of Range f.

Then, there exists $x\in R-\left\{-\frac{4}{3}\right\}$ such that. $y=f(x)$.

$$\Rightarrow y=\frac{4x}{3x+4}$$

$$\Rightarrow 3xy+4y=4x$$

$\Rightarrow x(3-3y)=4y$ let us define g: Range $f\rightarrow R-\left\{-\frac{4}{3}\right\}$ as $g(y)=\frac{4y}{4-3y}$.

$$\Rightarrow x = \frac{4y}{4-3y}$$

$$(gof)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$$

Now,

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12} = \frac{16x}{16} = x$$

$$\text{and,}(fog)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4+3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16x}{12y+16-12y} = \frac{16y}{16} = y$$

$$\therefore gof = I_{R-\left\{-\frac{4}{3}\right\}} \text{andfog} = I_{\text{Rangef}}$$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map g: Range $f \to R - \left\{-\frac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.