SpeedLabs

MATHS

CBSE 12th

TEEVRA EDUTECH PVT. LTD.

Continuity and Differentiability Exercise-5.3

SpeedLabs
Assisted Practice | Expert Guidance | Personalized Insights
"An Innovative Practice Methodology by IITians."

Q.1 Find $\frac{dy}{dx}$: $2x + 3y = \sin x$

Sol: The given relationship is $2x + 3y = \sin x$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin x)$

$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos x$

$\Rightarrow 2 + 3\frac{dy}{dx} = \cos x$

$\Rightarrow 3\frac{dy}{dx} = \cos x - 2$

$\therefore \frac{dy}{dx} = \frac{\cos x - 2}{3}$

Q.2 Find $\frac{dy}{dx}$: $2x + 3y = \sin y$

Sol: The given relationship is $2x + 3y = \sin y$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(2x + 3y) = \frac{d}{dx}(\sin y)$

$\Rightarrow \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \cos y \frac{dy}{dx}$[By using chain rule]

$\Rightarrow 2 = (\cos y - 3)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{2}{\cos y - 3}$

Q.3 Find $\frac{dy}{dx}$: $ax + by^2 = \cos y$

Sol: The given relationship is $ax + by^2 = \cos y$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(ax) + \frac{d}{dx}(by^2) = \frac{d}{dx}(\cos y)$

$\Rightarrow a + b\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos y)$ .... (1)

By using chain rule, we obtain $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ and $\frac{d}{dx}(\cos y) = -\sin y \frac{dy}{dx}$

From (1) and (2), we obtain

$a + b \times 2y\frac{dy}{dx} = -\sin y \frac{dy}{dx}$

$\Rightarrow (2by + \sin y)\frac{dy}{dx} = -a$

$\therefore \frac{dy}{dx} = \frac{-a}{2by + \sin y}$

Q.4 Find $\frac{dy}{dx}$: $xy + y^2 = \tan x + y$

Sol: The given relationship is $xy + y^2 = \tan x + y$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(xy + y^2) = \frac{d}{dx}(\tan x + y)$$

$$\Rightarrow \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$\Rightarrow \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$ [Using product rule and chain rule]

$$\Rightarrow y.1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{(x+2y-1)}$$

Q.5 Find $\frac{dy}{dx}$: $x^2 + xy + y^2 = 100$

Sol: The given relationship is $x^2 + xy + y^2 = 100$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx}(100)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = 0$$ [Derivative of constant function is 0]

$$\Rightarrow 2x + \left[y.\frac{d}{dx}(x) + x.\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$ [Using product rule and chain rule]

$$\Rightarrow 2x + y.1 + x.\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{2x+y}{x+2y}$$

Q.6 Find $\frac{dy}{dx}$: $x^3 + x^2y + xy^2 + y^3 = 81$

Sol: The given relationship is $x^3 + x^2y + xy^2 + y^3 = 81$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(x^3 + x^2y + xy^2 + y^3) = \frac{d}{dx}(81)$$

$$\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(y^3) = 0$$ [Derivative of constant function is 0]

$$\Rightarrow 3x^2 + \left[y.\frac{d}{dx}(x^2) + x^2\frac{dy}{dx}\right] + \left[y^2.\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 + \left[y.2x + x^2\frac{dy}{dx}\right] + \left[y^2.1 + x.2y.\frac{dy}{dx})\right] + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow (x^2 + 2xy + 3y^2)\frac{dy}{dx} + (3x^2 + 2xy + y^2) = 0$$

$$\therefore \frac{dy}{dx} = -\frac{(3x^2+2xy+y^2)}{(x^2+2xy+3y^2)}$$

Q.7 Find $\frac{dy}{dx}$: $\sin^2 y + \cos xy = \pi$

Sol: The given relationship is $\sin^2 y + \cos xy = \pi$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(\pi)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0 \qquad \ldots\ldots\ldots (1)$$

By using chain rule, we obtain

$$\frac{d}{dx}(\sin^2 y) = 2\sin y \frac{d}{dx}(\sin y) = 2\sin y \cos y \frac{dy}{dx} \qquad \ldots\ldots\ldots (2)$$

$$\Rightarrow \frac{d}{dx}(\cos xy) = -\sin xy \frac{d}{dx}(xy) = -\sin xy\left[y\frac{d}{dx}(x) + x\frac{dy}{dx}\right]$$

$$= -\sin xy\left[y.1 + x\frac{dy}{dx}\right] = -y.\sin xy - x\sin xy\frac{dy}{dx} \qquad \ldots\ldots\ldots (3)$$

From (1), (2), and (3), we obtain

$$\Rightarrow 2\sin y\cos y\frac{dy}{dx} - y\sin xy - x\sin xy\ \frac{dy}{dx} = 0$$

$$\Rightarrow (\sin 2y - x\sin xy)\frac{dy}{dx} = y\sin xy$$

$$\therefore \frac{dy}{dx} = -\frac{y\sin xy}{(\sin 2y - x\sin xy)}$$

Q.8 Find $\frac{dy}{dx}$: $\sin^2 x + \cos^2 y = 1$

Sol: The given relationship is $\sin^2 x + \cos^2 y = 1$

Differentiating this relationship with respect to x, we obtain

$$\frac{d}{dx}(\sin^2 x + \cos^2 y) = \frac{d}{dx}(1)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 y) = 0$$

$$\Rightarrow 2\sin x.\frac{d}{dx}(\sin x) + 2\cos y.\frac{d}{dx}(\cos y) = 0$$

$$\Rightarrow 2\sin x.\cos x + 2\cos y.(-\sin y).\frac{dy}{dx} = 0$$

$$\Rightarrow \sin 2x - \sin 2y\ \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\sin 2x}{\sin 2y}$$

Q.9 Find $\frac{dy}{dx}$: $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Sol: The given relationship is $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \sin y = \frac{2x}{1+x^2}$$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{2x}{1+x^2}\right)$ .... (1)

The function, $\frac{2x}{1+x^2}$, is of the form of $\frac{u}{v}$.

Therefore, by quotient rule, we obtain

$\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = \frac{(1+x^2)\frac{d}{dx}(2x) - 2x.\frac{d}{dx}(1+x^2)}{(1+x^2)}$

$\Rightarrow \frac{(1+x^2).\ 2 - 2x.\ [0+2x]}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$ .... (2)

Also, $\sin y = \frac{2x}{1+x^2}$

$\Rightarrow 2\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{2x}{1+x^2}\right)^2} = \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}$

$= \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}} = \frac{1-x^2}{1+x^2}$ .... (3)

From (1), (2), and (3), we obtain

$\frac{1-x^2}{1+x^2} \times \frac{dy}{dx} = \frac{2(1-x^2)^2}{(1+x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{1+x^2}$

Q.10 Find $\frac{dy}{dx}$: $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

Sol: The given relationship is $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$\Rightarrow \tan y = \left(\frac{3x-x^3}{1-3x^2}\right)$ .... (1)

It is known that, $\tan y = \frac{3\tan\frac{y}{3} - \tan^3\frac{y}{3}}{1-3\tan^2\frac{y}{3}}$ .... (2)

Comparing equations (1) and (2), we obtain

$x = \tan\frac{y}{3}$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{3}\right)$

$\Rightarrow 1 = \sec^2\frac{y}{3}.\frac{d}{dx}\left(\frac{y}{3}\right)$

$\Rightarrow 1 = \sec^2\frac{y}{3}.\frac{1}{3}\frac{dy}{dx}$

$\Rightarrow 1 = \frac{3}{\sec^2\frac{y}{3}} = \frac{3}{1+\tan^2\frac{y}{3}}$

$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$

Q.11 Find $\frac{dy}{dx}$: $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Sol: The given relationship is $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \cos y = \left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}} = \frac{1-x^2}{1+x^2}$

On comparing L.H.S. and R.H.S. of the above relationship, we obtain

$x = \tan\frac{y}{2}$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(x) = \frac{d}{dx}\left(\tan\frac{y}{2}\right)$

$\Rightarrow \sec^2\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right) = \frac{d}{dx}(x)$

$\Rightarrow \sec^2\frac{y}{2} \times \frac{1}{2}\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\sec^2\frac{y}{2}} = \frac{2}{1+\tan^2\frac{y}{2}}$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$

Q.12 Find $\frac{dy}{dx}$: $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Sol: The given relationship is $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \sin y = \left(\frac{1-x^2}{1+x^2}\right)$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$ ...... (1)

By using chain rule, we obtain

$\frac{d}{dx}(\sin y) = \cos y.\frac{dy}{dx}$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$

$= \sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}} = \sqrt{\frac{4x^2}{(1+x^2)^2}} = \frac{2x}{1+x^2}$

$\therefore \frac{d}{dx}(\sin y) = \frac{2x}{1+x^2}\frac{dy}{dx}$ ...... (2)

$\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right) = \frac{(1+x^2).(1-x^2)'-(1-x^2).(1+x^2)'}{(1+x^2)^2}$ [Using quotient rule]

$= \frac{(1+x^2).(1-x^2)-(1-x^2).(1+x^2)}{(1+x^2)^2}$

$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}$

$= \frac{-4x}{(1+x^2)^2}$ ...... (3)

From (1), (2), and (3), we obtain

$\frac{2x}{1+x^2}\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{2x}{1+x^2}$

Alternate method

$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \sin y = \frac{1-x^2}{1+x^2}$

$\Rightarrow (1+x^2)\sin y = 1 - x^2$

$\Rightarrow (1+\sin y)x^2 = 1 - \sin y$

$\Rightarrow x^2 = \frac{1-\sin y}{1+\sin y}$

$\Rightarrow x^2 = \frac{\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)^2}{\left(\cos\frac{y}{2} + \sin\frac{y}{2}\right)^2}$

$\Rightarrow x = \frac{\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)}{\left(\cos\frac{y}{2} + \sin\frac{y}{2}\right)}$

$\Rightarrow x = \frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}$

$\Rightarrow x = \tan\left(\frac{\pi}{4} - \frac{y}{2}\right)$

Differentiating this relationship with respect to x, we obtain

$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}\left[\tan\left(\frac{\pi}{4} - \frac{y}{2}\right)\right]$

$\Rightarrow 1 = \sec^2\left(\frac{\pi}{4} - \frac{y}{2}\right).\frac{d}{dx}\left(\frac{\pi}{4} - \frac{y}{2}\right)$

$\Rightarrow 1\left[1 + \tan^2\left(\frac{\pi}{4} - \frac{y}{2}\right)\right].\left(-\frac{1}{2}\frac{dy}{dx}\right)$

$\Rightarrow 1 = (1+x^2).\left(-\frac{1}{2}\frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$

Q.13 Find $\frac{dy}{dx}$: $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Sol: The given relationship is $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow \cos y = \frac{2x}{1+x^2}$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(\cos y) = \frac{d}{dx}.\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow -\sin y.\frac{dy}{dx} = \frac{(1+x^2).\frac{d}{dx}(2x) - 2x.\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$

$\Rightarrow \sqrt{1-\cos^2 y}\frac{dy}{dx} = \frac{(1+x^2)\times 2 - 2x.2x}{(1+x^2)^2}$

$\Rightarrow \left[\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}\right]\frac{dy}{dx} = -\left[\frac{2(1-x^2)}{(1+x^2)^2}\right]$

$\Rightarrow \sqrt{\frac{(1+x^2)^2-4x^2}{(1+x^2)^2}}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$\Rightarrow \sqrt{\frac{(1-x^2)^2}{(1+x^2)^2}}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$\Rightarrow \frac{1-x^2}{1+x^2}\frac{dy}{dx} = \frac{-2(1-x^2)}{(1+x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$

Q.14 Find $\frac{dy}{dx}$: $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

Sol: The given relationship is $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \sin y = 2x\sqrt{1-x^2}$

Differentiating this relationship with respect to x, we obtain

$(\cos y)\frac{dy}{dx} = 2\left[x\frac{d}{dx}\left(\sqrt{1-x^2}\right) + \sqrt{1-x^2}\frac{dx}{dx}\right]$

$\Rightarrow \sqrt{1-\sin^2 y}\frac{dy}{dx} = 2\left[\frac{x}{2}.-\frac{2x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$

$\Rightarrow \left[\sqrt{1-(2x\sqrt{(1-x^2)^2}}\right]\frac{dy}{dx} = 2\left[\frac{-x^2+1-x^2}{\sqrt{1-x^2}}\right]$

$\Rightarrow \sqrt{1-4x^2(1-x^2)}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$

$\Rightarrow \sqrt{(1-2x^2)^2}\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$

$\Rightarrow (1-2x^2)\frac{dy}{dx} = 2\left[\frac{1-2x^2}{\sqrt{1-x^2}}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$

Q.15 Find $\frac{dy}{dx}$: $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Sol: The given relationship is $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$

$\Rightarrow \ \sec y = \frac{1}{2x^2-1}$

$\Rightarrow \ \cos y = 2x^2 - 1$

$\Rightarrow \ 2x^2 = \cos y + 1$

$\Rightarrow \ 2x^2 = 2\cos^2\frac{y}{2}$

$\Rightarrow \ x = \cos\frac{y}{2}$

Differentiating this relationship with respect to x, we obtain

$\frac{d}{dx}(x) = \frac{d}{dx}\left(\cos\frac{y}{2}\right)$

$\Rightarrow 1 = -\sin\frac{y}{2}.\frac{d}{dx}\left(\frac{y}{2}\right)$

$\Rightarrow \frac{-1}{\sin\frac{y}{2}.} = \frac{1}{2}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{-2}{\sin\frac{y}{2}.} = \frac{-2}{\sqrt{1-\cos^2\frac{y}{2}.}}$

$\Rightarrow \ \frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$